Linear Algebra
(Part-2)

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Four Fundamental Subspaces

Let A be an $m \times n$ matrix of real entries.

1. The **column space** of A, $\mathcal{C}(A)$ is the space spanned by columns of A. That is,
$$\mathcal{C}(A) := \{Ax : x \in \mathbb{R}^n\}.$$
2. The **null space** of A, $\mathcal{N}(A)$ is the solution set of the matrix equation $Ax = 0$. That is,
$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$
3. The **row space** of A, is the space spanned by rows of A. It is same as the column space of A^T. That is,
$$\mathcal{R}(A^T) := \{y^T A : y \in \mathbb{R}^m\}.$$
4. The **left nullspace** of A is the nullspace of A^T. That is,
$$\mathcal{N}(A^T) := \{y \in \mathbb{R}^m : y^T A = 0\}.$$

Rank, Row rank, Column rank

- $dim(C(A)) = dim(C(A^T)) = Rank(A) = r$
- Prove: **Row rank = Column rank**
  Question: If the rows of $A_{n \times n}$ are independent how many independent columns does it have?

The Row Space

- Use Gaussian elimination to transform $[A|b]$ into echelon form $[U|c]$. Transforming to the echelon matrix means that we are taking linear combinations of the rows of a matrix A to come up with the matrix U.
- We could reverse the process and get back to A, by row operations again. Therefore, the row space of A equals the row space of U.
- If 2 matrices are row equivalent, then their row spaces are the same.
- The nonzero rows (rewritten as column vectors) of the matrix U form a **BASIS** for the row space.

Example for a basis of the row space of A

$$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is the echelon matrix of } A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}.$$

A basis for the row space of A is the set of non-zero rows of U (rewritten as column vectors). In our example, a basis for the row space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Notice the number of elements in the basis, which is the number of non-zero rows in U, is just the number of pivots.

The Column Space

- The column space of a matrix consists of **ALL** linear combinations of the columns of the matrix.
- Reduce A to the echelon matrix U, by row operations.
- Find the pivot variables $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ where $r = \text{rank}(A)$ is the total number of pivots. A basis for the column space of A is the set of columns $i_1, i_2, \ldots, i_r$ in the original matrix A.
- That is, the columns of the original matrix corresponding to those columns containing **PIVOTS** form a **BASIS** for the column space of the matrix.
- Notice the number of elements in the basis, which is the number of non-zero rows in U, is just the number of pivots. This means **the column space and the row space of a matrix always have the same dimension**.

Example for a basis of the column space of A

$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the echelon matrix of $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}$.

Since the pivot variables of A are x_1 and x_4, a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right\}.$$

Caution. We had to choose columns from the original matrix A, because Gaussian elimination changes the column picture at each step.

To see this in our example, just note that the columns of U all have a zero in the third entry: if these spanned the column space of A then every column of A would also have to have zero in the third entry–which is false!

Nullspace

- The system $Ax = 0$ is reduced to $Ux = 0$, where U is an echelon matrix, and this process is reversible.
- The nullspace of A is the same as the nullspace of U.
- Find the free variables $x_{j_1}, x_{j_2}, \ldots, x_{j_{n-r}}$ where $n - r$ is the number of columns of A without a pivot. There is a basis for the nullspace of A, with one vector associated to each free variable. Taking each free variable one at a time, set that free variable as 1 and the other free variables as 0. Then solve $Ux = 0$ for the vector x with this choice and put x in the basis. Repeat for each free variable.
- The $n - r$ "special solutions" to $Ux = 0$ provides a **BASIS** for the nullspace.

Example for a basis of the null space of A

$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the echelon matrix of $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}$.

The free variables are x_2 and x_3. Taking $x_2 = 1, x_3 = 0$ and $x_2 = 0, x_3 = 1$ and solve $Ux = 0$ for each choice. A basis for the nullspace of is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Left Nullspace

- Use Gaussian elimination to transform $[A|b]$ into echelon form $[U|c]$.
- A basis for the left nullspace of A has $m - r$ vectors, which is the number of zero rows in U.
- For each zero row, put a vector in the basis whose entries are the coefficients of the vector b in the entry of c corresponding to the zero row.

Example for a basis of the left null space of A

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix} = [U \mid c].$$

There is one zero row of U. As an equation the row represents $0 = -2b_1 - b_2 + b_3$ (note we listed the b_i's in order). Thus a basis for the left nullspace is

$$\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

In general, the number of elements in this basis will equal the number of zero rows.

Dimensions of Four Fundamental Subspaces

Let A be $m \times n$ with rank r. Using the bases above, we observe the following:

- dim $R(A) =$ dim $R(A^T) = r$ (the number of pivots).
- dim $N(A) = n - r$ (the number of free columns).
- dim $N(A^T) = m - r$ (the number of zero rows).

Notice that the column space and the row space of a matrix have the same dimension (even though the vectors in each subspace live in a different ambient space, $\mathbb{R}^m$ and $\mathbb{R}^n$.)

The nullspace and row space live in $\mathbb{R}^n$; the left nullspace and column space live in $\mathbb{R}^m$.

Fundamental Theorem of Linear Algebra (Part I)

There is an important relationship between the dimensions of the subspaces in each of these pairs of subspaces, which is shown by the following theorem.

Theorem 1.

Let A be $m \times n$ with rank r. Then

- $dim\ R(A^T) + dim\ N(A) = n.$
- $dim\ R(A) + dim\ N(A^T) = m.$

We call the dimension of $N(A)$, **nullity** of A.

The first equation is also called the **rank-nullity law, or, rank-nullity theorem** because rank $A = \dim\ R(A^T)$.

To Remember Their Dimensions !

To remember what are the dimensions of the four fundamental subspaces, it is best just to think about where the bases for each subspace comes from :

- The bases for the column space and row space come from the pivots, so the dimension of each of these subspaces is the rank of the matrix.
- The basis for the nullspace comes from the free columns, so the dimension of the nullspace is the number of free columns.
- The basis for the left nullspace is obtained from the zero rows, so the dimension of the left nullspace is the number of zero rows.

Figure : Nullspace and Row Space in $\mathbb{R}^n$

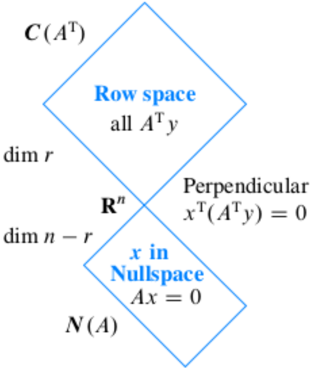
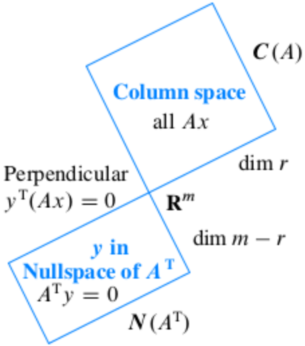


Figure : Left Nullspace and Column Space in $\mathbb{R}^m$



Exercises

Exercises 2.

1. $w = (3, 1, 2) \in \mathbb{R}^3$. *What is* $W = span(w)$*?*
2. $span\{(1, 0, 0), (0, 1, 0)\}$ *&* $span\{(1, 0, 0), (0, 1, 0), (3, 5, 0)\}$*?*
   $span\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *&* $span\{(3, 1, 2), (7, 1, 0), (0, 0, 1)\}$*?*
3. $span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} \right\}$*? Does it span all* 2×2 *matrices with real entries?*
4. *What is* $span\{1, x, x^2\}$*?*
5. *If* $A = [a_1 a_2 \dots a_n]$*, then* $span\{a_1, a_2, \dots, a_n\}$*?*
6. *Is it possible* $\alpha_1 w_1 + \cdots + \alpha_l w_l = \beta_1 w_1 + \cdots + \beta_l w_l$*, for a fixed* $(\alpha_1, \cdots, \alpha_l) \neq (\beta_1, \cdots, \beta_l)$*? What does it mean in terms of the solutions of a linear system?*
7. *Question:* *Is* $W = span(S)$ *a subspace of* V*? Prove!*
8. *Consider* $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. *Span*$\{e_1, \cdots, e_n\} = \mathbb{R}^n$*. Is there any other* n *vectors that can span* $\mathbb{R}^n$*?*

Exercises 3.

1. $\mathbb{R}^n = span\{e_1, \cdots, e_n\}$. *Standard basis of* $\mathbb{R}^n$.
2. $B_1 = \{1,0),(0,1)\}, B_2 = \{(1,2),(2,5)\},\ \ B_3 = \{(2,0),(0,3),(5,6)\}$. *Are they basis for* $\mathbb{R}^2$*?*
3. *Find the basis of* $C(U)$, *where* $U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
4. *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. *Do the columns of* A *form basis for* $\mathbb{R}^4$*?*
5. *Find a basis for* $W = span\{2x^3 + 4, x^2 - 7, x^3 + x^2, 2x^2 + 2\}$.
6. *Say true or false : Every vector space can have many bases.*
7. *When do columns of a matrix* $A_{n \times n}$ *form a basis for* $\mathbb{R}^n$*?*
8. *Prove that the set of all columns of any* $n \times n$ *invertible matrix* ($a_{ij} \in \mathbb{R}$) *form a basis for* $\mathbb{R}^n$.
9. *Prove that if a basis is given for a vector space, then each vector in the space has unique expression as a linear combination of elements in that basis.*

Exercises 4.

1. *Show that any two bases for a vector space V contain same number of elements from V. This number, shared by all bases, represent the number of "degrees of freedom" of the space.*
2. *Any linearly independent set in V can be extended to a basis by adding more vectors if necessary. (Any vectors can be added?) Similarly, any spanning set S in V can be reduced to a basis by discarding unnecessary vectors. (What are those unnecessary vectors?)*
3. *Is dimension always a finite number? Give some simple examples of vector space with infinite dimension.*

Left and right inverse

- Let A be an $m \times n$ matrix, then the following are equivalent:
    1. A has a right inverse, i.e., there exists an $n \times m$ matrix C such that $AC = I$.
    2. The system $Ax = b$ has at least one solution x for each $b \in \mathbb{R}^m$.
    3. The $C(A) = \mathbb{R}^m$.

  Also, any one of (1), (2), or (3) implies that $m \leq n$. (Prove).

- Let A be an $m \times n$ matrix, then the following are equivalent:
    1. A has a left inverse, that is, there exists an $n \times m$ matrix B such that $BA = I$.
    2. The system $Ax = b$ has at most one solution x for each $b \in \mathbb{R}^m$
    3. The columns of A are linearly independent.

  Also, any one of (1), (2), or (3) implies that $m \geq n$. (Prove).

Invertibility condition

- Can a rectangular matrix have both existence and uniqueness?
- A square matrix has a left-inverse, iff, it has a right-inverse. Further, $B = C = A^{-1}$. Also full rank means $r = m = n$..
- Best left and right inverses (if they exist): $B = (A^T A)^{-1} A^T$ and $C = A^T (AA^T)^{-1}$ But what about the inverse of $A^T A$?

Rank 1 matrix

Every matrix of rank $= 1$ has the simple form $A = uv^T =$ column matrix times row matrix.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

Linear Transformation

Definition 1.

Let V and U be two vector spaces (over $\mathbb{R}$). A function $T : V \to U$ is a Linear Transformation if it satisfies the following

1. $T(x + y) = T(x) + T(y)$, $\forall x, y \in V$
2. $T(\alpha x) = \alpha T(x)$, $\forall \alpha \in \mathbb{R}$ and $\forall x \in V$

Examples

1. $T : \mathbb{R} \to \mathbb{R}$ defined by
   - $T(x) = 0$, $\forall x \in \mathbb{R}$
   - $T(x) = ax$, $\forall x \in \mathbb{R}$ & for some fixed constant $a \in \mathbb{R}$.
   - Any other expression?
2. $T : \mathbb{R} \to \mathbb{R}^2$ defined by
   - $T(x) = 0$, $\forall x$
   - $T(x) = (x, 2x)$, $\forall x$
3. $T : \mathbb{R}^2 \to \mathbb{R}$ defined by
   - $T((x, y)) = x + y$, $\forall (x, y) \in \mathbb{R}^2$
   - $T(x, y) = x$, $\forall (x, y) \in \mathbb{R}^2$
4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
   - $T(x, y) = (0, 0)$, $\forall (x, y) \in \mathbb{R}^2$
   - $T(x, y) = (x, y)$, $\forall (x, y) \in \mathbb{R}^2$
   - $T(x, y) = (y, x)$, $\forall (x, y) \in \mathbb{R}^2$
   - $T(x, y) = (0, x)$, $\forall (x, y) \in \mathbb{R}^2$
   - $T(x, y) = (-y, x)$, $\forall (x, y) \in \mathbb{R}^2$
   - $T(x, y) = (x + y, y - x)$, $\forall (x, y) \in \mathbb{R}^2$
   - $T(x, y) = 2(x, y)$, $\forall (x, y) \in \mathbb{R}^2$

Examples

1. $T : \mathbb{R}^3 \to \mathbb{R}$ defined by
   - $T((x, y, z)) = x + y - z$, $\forall (x, y, z) \in \mathbb{R}^3$
   - $T(x, y, z) = y$, $\forall (x, y, z) \in \mathbb{R}^3$
2. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
   - $T(x, y, z) = (y, z, x)$, $\forall (x, y, z) \in \mathbb{R}^3$
   - $T(x, y, z) = (x + y, y + z, z)$, $\forall (x, y, z) \in \mathbb{R}^3$
3. Let $\mathbb{P}_n$ be the set of all polynomials with degree $\leq n$ with real coefficients.
   - $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by $T(p(x)) = p'(x)$.
   - $T : \mathbb{P}_n \to \mathbb{P}_{n+1}$ defined by $T(p(x)) = \int_0^x p(t)dt$.
   - $T : \mathbb{P}_n \to \mathbb{P}_{n+1}$ defined by $T(p(x)) = (2 + x)p(x)$.
4. Let $\mathbb{R}^\infty$ be the set of all real sequences. $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by
   - $T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$.
   - $T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$.
   - $T(x_1, x_2, x_3, \ldots) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots)$.

Not a linear map?

1. $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1 \ \forall x$
2. $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = xy \ \forall x, y$
3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, 1 - y) \ \forall x, y$
4. $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by $T(p(t)) = p(t)^2$.
5. $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by $T(p(t)) = \sin p(t)$

Exercises

1. Let $T : V \to U$ be a linear map. What is $T(0)$?
2. Let $range(T) = T(V)$ and kernel(T)=ker(T)=$\{x \in V : T(x) = 0\}$. Then find the following for all the linear maps (transformations) mentioned above.
   - $range(T)$
   - $ker(T)$
   - their dimensions?
   - a matrix A such that $T(x) = Ax$
   - Compare the $C(A)$ and $range(T)$; $N(A)$ and $ker(T)$
3. Give 5 examples of linear map and not linear map on infinite dimensional spaces.
4. If A is a matrix of a linear map T, then what is the matrix for T^2?
5. Is composition of two linear maps a linear map?

Transformations represented by matrices

What is the linear map which takes $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}2\\3\\4\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}4\\6\\8\end{bmatrix}$?

Importance of basis

- Difference between basis & ordered basis?
- Two bases of $\mathbb{R}^2$: $B_1 = \{(1,0),(0,1)\}$ & $B_2 = \{(1,1),(3,5)\}$
- Express $(5,-2)$ in terms of two ordered bases B_1 & B_2
- Express $(5,-2,1)$ in terms of two ordered bases of $\mathbb{R}^3$:
  $B_1 = (1,0,0),(0,1,0),(0,0,1)$ & $B_2 = \{(1,1,0),(3,5,1),(2,4,2)\}$

Matrix of a linear map

- Find a matrix for the linear map defined as $T(x, y) = (x, 0, y)$ with respect to the standard basis in the domain and co-domain.
- Remember earlier slide? What is the LT which takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$?

Theorem 2.

Let T be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then there is an $m \times n$ matrix A such that $T(x) = Ax$, for all $x \in \mathbb{R}^n$.

Theorem 3.

A linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is completely determined by the images of the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$.

Matrix of a linear map w.r.t other bases

Find a matrix for the linear map defined as $T(x, y) = (x, 0, y)$.

1. with respect to the basis (ordered) $\{(1, 0), (1, 1)\}$ in the domain and $\{(1, 0, 0), (0, 1, 0), (1, 2, 1)\}$ in the co-domain.
2. with respect to the basis (ordered) $\{(1, 0), (1, 1)\}$ in the domain and $\{(1, 1, 1), (1, 2, 1), (0, 0, 1)\}$ in the co-domain.

Theorem 4.

Suppose the vectors $B_1 = \{x_1, x_2, \ldots, x_n\}$ is a basis for the space V and $B_2 = \{y_1, y_2, \ldots, y_m\}$ is a basis for U. Then, for each linear map $T : V \to U$, there is a matrix A with respect to B_1 and B_2. Also, entries of A can be obtained from

$$T(x_j) = Ax_j = a_{1j}y_1 + a_{2j}y_1 + \cdots + a_{mj}y_m, \; j = 1, 2, \ldots, n$$

Remarks: For any $x \in V$, $\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}$ where

$$x = \sum_1^n \alpha_i x_i, \quad T(x) = \sum_1^m \beta_j y_j, \quad (a_{ij}) = [T]_{B_1}^{B_2}$$

Examples

Example 5.

Find the matrix for the linear map $T : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $T(f) = f'$ with respect to the bases $B_1 = B_2 = \{1, x, x^2, x^3\}$ in domain as well as the codomain.

Example 6.

Find the matrix for the linear map $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by $T(f(x)) = \int_0^x f(t)dt$ with respect to the bases $B_1 = \{1, x, x^2\}$ in domain and $B_2 = \{1, x, x^2, x^3\}$ in the codomain.

Matrices of special linear maps

	Matrix	Properties
Rotation through 90 deg $T(x,y) = (-y,x)$	$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$Q^4 = Q$
Rotation through θ deg	$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	1. $Q_\theta Q_{-\theta} = I$ 2. $Q_\theta^2 = Q_{2\theta}$ 3. $Q_\theta Q_\phi = Q_{\theta+\phi}$
Projection onto x axis $T(x,y) = (x,0)$	$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	
Projection onto θ line	$P_\theta = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$	1. No inverse 2. θ-line map to θ-line 3. $P_\theta^2 = P_\theta$ 4. $N(P_\theta)=$ the line $\perp$ to θ-line
Reflection w.r.t. $y = x$ $T(x,y) = (y,x)$	$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	
Reflection w.r.t. θ line	$H_\theta = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}$	1. $H_\theta = 2P_\theta - I$ 2. $H_\theta^2 = I$ 3. $H_\theta^{-1} = H_\theta$

References

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